# On Unification of Gravity and Electromagnetism and the Absence of Magnetic Monopoles

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**Summary. -** It was demonstrated earlier that the factorization of Einstein's second-rank, symmetric tensor field equations, by removing the space and time-reflection elements from the underlying covariance group, yields a 16-component quaternion metrical field equation. The 16 equations were then re-expressed in a second-rank tensor form, breaking up into $10 \oplus 6$ equations. The 10 equations are in one-to-one correspondence with Einstein's symmetric tensor equations for gravitation; they are even under reflections. The remaining 6 equations are in antisymmetric tensor form; they are odd under reflections. Taking the covariant divergence of the latter equations then puts them into one-to-one correspondence with the structure of Maxwell's equations for electromagnetism. What was shown earlier was that the four of Maxwell's equations with sources followed. What is shown in this paper is that the other four of Maxwell's equations that are source-free also follow, thus indicating the absence of magnetic monopoles from this theory, as in the usual Maxwell formalism. This shows that the factorization of Einstein's field equations to the quaternion form fully unifies the gravitational and electromagnetic manifestations of interacting charged matter.
PACS. 04.90.+e General relativity and gravitation
14.80.Hv Magnetic monopoles

## 1. Introduction

In earlier papers it was argued that the symmetry group which underlies general relativity theory – *the Einstein group* - is a 16-parameter Lie group. That is, this is a group of continuous, analytic transformations, characterized by the 16 essential parameters, $\partial x^{\mu'}/\partial x^{\nu}$, distributed continuously and analytically in space-time. ([1])

The symmetry group that underlies the covariance of Einstein's field equations includes the discrete reflections in space and time, which are not in the covariance requirement of relativity theory – thus, the representations of the group that underlies Einstein's tensor equations are reducible. When the reflections are removed from the group, its representations change to the irreducible form and Einstein's equations factorize to quaternion formalism.

The new metrical field is the four-vector, $q^{\mu}(x)$, whose components are each quaternion-valued, rather than being real number-valued. Thus, the factorized metrical field has 4 x 4 = 16 independent components, rather than 10 (as it should, in accordance with the 16-parameter Lie group). The factorized form of the metrical field equations were then shown to be the 16 independent field relations as follows: ([2])

(1a) $$(1/4)(K_{\rho\lambda}q^{\lambda} + q^{\lambda} K_{\rho\lambda}^{+}) + (1/8) Rq_{\rho} = kT_{\rho}$$

The quaternion equation conjugate to this one (its space or time reflection) is:

(1b) $$-(1/4)(K_{\rho\lambda}^{+}q^{\lambda} + q^{\lambda} K_{\rho\lambda}) + (1/8) Rq_{\rho} = kT_{\rho}$$

where the 'spin curvature' tensor $K_{\rho\lambda}$ will be defined below and

(1c) $$R = (1/2)(K_{\rho\lambda}q^{\lambda} q^{\rho} - q^{\rho} q^{\lambda} K_{\rho\lambda} + q^{\lambda} K_{\rho\lambda}^{+}q^{\rho} - q^{\rho} K_{\rho\lambda}^{+}q^{\lambda})$$

is the Riemann scalar curvature in quaternion form. With this new formalism, the invariant differential of the curved space-time is a quaternion-valued scalar

(2) $$ds = q^{\mu}(x)dx_{\mu}$$

The geometric property of $q^{\mu}$ is that of a four-vector field; its algebraic property is that of a second-rank spinor of the form, $\psi \otimes \psi^{*}$. Together with the vanishing of the second covariant derivatives of the quaternion metrical field, the difference $q^{\mu}{}_{;\lambda;\rho} - q^{\mu}{}_{;\rho;\lambda} = 0$ then leads to the form of the Riemann curvature tensor, $R_{\mu\nu\lambda\rho}$, the Ricci tensor, $R_{\mu\nu}$, and the Riemann scalar curvature field R (as shown above in eq. 1c), as functions of the quaternion field $q^{\lambda}$ and the spin curvature field, $K_{\lambda\rho}$. ([2])

The variational calculus applied to $R\sqrt{-g}$ as the metrical field Lagrangian density then yields the factorized form of the field equations (1ab). Multiplying eq. (1a) on the right with the conjugated solution $q_\gamma$ and the conjugated equation (1b) on the left with the solution $q_\gamma$, one then forms a second rank tensor equation that is neither symmetric nor antisymmetric. *Adding* these two equations then yields a set of symmetric tensor field equations that are in one-to-one correspondence with the Einstein tensor field equations, ([3])

$$R_{\mu\nu} \ - \ (1/2)\, g_{\mu\nu}\, R \ = \ kT_{\mu\nu} \tag{3}$$

*Subtracting* the two equations then yields a set of antisymmetric second-rank tensor equations. The former is even under reflections in space or time and the latter are odd under these reflections. Thus, the 6 antisymmetric tensor equations can be put into one-to-one correspondence with the structure of Maxwell's field equations by taking their covariant divergence.

In this case, it has been shown that the structure of those of Maxwell's field equations for electromagnetism, with the source terms on the right-hand side, follows:

$$F_{\rho\gamma}{}^{;\rho} \ = \ (4\pi/c) j_\gamma \tag{3}$$

These are four out of the eight Maxwell field equations. In eq. (4),

(5a) $F_{\rho\gamma} \ = \ Q[(1/4)(K_{\rho\gamma}\, q^\lambda\, q_\gamma \ + \ q_\gamma\, q^\lambda\, K_{\rho\lambda} \ + \ q^\lambda\, K_{\rho\lambda}{}^{+} q_\gamma \ + \ q_\gamma\, K_{\rho\lambda}{}^{+} q^\lambda\,) \ +$
$(1/8)(q_\rho q_\gamma \ - \ q_\gamma q_\rho\,)R]$ .

$$j_\gamma \ = \ (cQk'/4\pi\,)(T_\rho{}^{;\rho} q_\gamma \ - \ q_\gamma\, T_\rho{}^{;\rho}) \tag{5b}$$

where Q is an arbitrary constant with the dimension of electric charge, inserted on both sides eq. (4), in order to give the correct units for this equation.

The spin-curvature tensor $K_{\rho\lambda}$ is defined in terms of the second covariant derivatives of a first-rank spinor field $\psi$ as follows:

$$\psi_{;\lambda;\rho} \ - \ \psi_{;\rho;\lambda} \ = \ K_{\lambda\rho}\psi \tag{6}$$

where

(7) $$K_{\lambda\rho} = \partial_\lambda\Omega_\rho - \partial_\rho\Omega_\lambda + \Omega_\lambda\Omega_\rho - \Omega_\rho\Omega_\lambda = -K_{\rho\lambda}$$

and $\Omega_\rho$ is the spin-affine connection field, defined in terms of the covariant derivative of a first-rank spinor field in a curved space-time, as follows:

$$\psi_{;\rho} = \partial_\rho\psi + \Omega_\rho\psi.$$

From the vanishing of the covariant derivatives of the quaternion metrical field, the explicit form of the spin-affine connection field is as follows:

(8) $$\Omega_\rho = (1/4)(\partial_\rho q^\mu + \Gamma_{\tau\rho}{}^\mu q^\tau)q_\mu = -(1/4)q_\mu(\partial_\rho q^\mu + \Gamma_{\tau\rho}{}^\mu q^\tau)$$

where $\Gamma_{\tau\mu}{}^\rho = \Gamma_{\mu\tau}{}^\rho$ are the ordinary affine connection terms.

## 2. The Remaining Maxwell's Equations and the Absence of Magnetic Monopoles

While the four of Maxwell's field equations (4) were demonstrated to follow from the factorization of Einstein's field equations, the remaining four that have no source terms were not explicitly shown before. But they indeed do follow from the generalized form of the tensor $F_{\rho\lambda}$ shown in eq. (5a). This is because of the fact that the antisymmetric spin curvature tensor $K_{\rho\lambda} = -K_{\lambda\rho}$ is the four-dimensional curl of a four vector $\Omega_\rho$, as we see in eq. (7). That is to say, the four of Maxwell's equations,

(9) $$F_{[\mu\nu;\lambda]} \equiv F_{\mu\nu;\lambda} + F_{\lambda\mu;\nu} + F_{\nu\lambda;\mu} = 0$$

in the ordinary form of the theory, have zero on the right because the tensor $F_{\mu\nu}$ is the four-dimensional curl of a four vector – the four-potential $A_\mu$ (subject to the Lorentz gauge). The same conclusion follows in this geometrical derivation of the Maxwell tensor, shown in eq. (5a), as we see below.

Since the covariant derivatives of the quaternion fields $q^\lambda$ are zero, it follows that the left-hand side of eq. (9), with (5a) and (1c) defining $F_{\mu\nu}$, depends on a sum of terms, each proportional to the cyclic sum $K_{[\rho\gamma;\lambda]}$. But just as in the case of the Maxwell tensor of the standard theory, this must vanish because $K_{\rho\gamma}$ is the four-dimensional curl of a four-vector – the spin-affine connection field, $\Omega_\rho$, as we see in eq. (7).

It should be noted that in configuration space, $\Omega_\rho$ is a four-vector, as we see in its expression in eq. (8). It is not covariant in spinor space (it cannot be so if it is the term that adds to the ordinary derivative of a spinor in order to make the derivative of $\psi$ in the curved space-time covariant, according to its definition, $\psi_{;\rho} = \partial_\rho\psi + \Omega_\rho\psi$). However, the vanishing of the cyclic sum $K_{[\rho\gamma;\lambda]} = 0$ only depends on the vanishing of the four-dimensional curl of the four-vector $\Omega_\rho$ in configuration space. Thus we see that because of the dependence of the electromagnetic tensor $F_{\rho\gamma}$ on the spin curvature tensor $K_{\rho\lambda}$ as shown in eq. (5a) and (1c), it follows that cyclic sum $F_{[\mu\nu;\lambda]} = 0$. These are the four of Maxwell's field equations that have no source terms on the right. Thus, there are no magnetic monopoles predicted in this theory, as in the standard form of the Maxwell field formalism, since a nonzero magnetic monopole source term would, of course, indicate a nonvanishing cyclic sum. This result, together with the four source-dependent equations derived here, $F_{\rho\nu}{}^{;\nu} = (4\pi/c)j_\rho$ ,then constitutes a one-to-one correspondence between the expression of the form of the electromagnetic field equations of the quaternion metrical formulation and the standard Maxwell formalism – including the lack of magnetic monopoles.

Thus we see that the quaternion factorization of the Einstein field equations do indeed yield a *unified field theory* wherein the 16-component metrical field $q^\mu$ entails both the gravitational and the electromagnetic manifestations of interacting matter.

I thank Professor C. Isham, of the Theoretical Physics Group, Imperial College, London, for his kind hospitality during the fall, 1997, when this note was prepared, and Professor T. Kibble for the question that led to the clarification presented in this note.

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