Motion in Einstein's Theory of Relativity and Comparisons with Classical Views

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Mendel Sachs

Department of Physics

University at Buffalo, State University of New York

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Symbols: p.6, 7, 8, 9: Δ = cap delta

p. 7: φ = l.c. phi

p. 12, 13: Σ = cap sigma, μ= l.c. mu, ν= l.c. nu

ABSTRACT

In everyday terms, as in Newtonian physics, 'motion' refers to a change of the position of an object in space with respect to a time change. That is to say, a continuous time change parameterizes a continuous spatial change along the trajectory of the object in motion. But in the later development of physics, particularly in the continuous field view that comes with Einstein's theory of relativity, 'motion' is a general type of change, not necessarily with respect to a time change. These differences in the meaning of the motion of matter in the classical atomistic view and the continuous field view will be addressed in this paper. Also discussed will be the question as to whether or not the 'motion' of matter is an absolute, *objective* quality of matter, or rather a *subjective* quality, dependent on the reference frame of the observer.

Introduction

The concept of 'motion' has an interesting history in physics, from ancient Greece to the Renaissance period of Descartes and Galileo, thence to the paradigm change in the twentieth century ideas that evolved with Einstein's theory of relativity.

Aristotle[1]

In ancient Greece, in Aristotle's teachings, motion was an absolute quality of matter, along with its shape, color and weight. He believed that something 'moves' because of where it is in space. At the center of space – at the Earth - no motion occurs, it is at rest.

Aristotle believed that the logical purpose of space is that it is there to be occupied by matter. This view also led Aristotle to the view that the space of the universe must be finite because he 'saw' that the quantity of matter in the universe is finite.

The actual motion, in his view, proceeds in terms of a series of discrete accidents. Indeed, the reason that the matter moves where it is is because it absorbs that motion in an accident from a preceding motion, which in turn, absorbed its motion from a still earlier motion, and so on, *ad infinitum*. Aristotle then concluded that 'time', *per se*, is a measure of the motion of matter and that there is no beginning of the universe.

Later scholars and theologians disputed this idea. They believed, rather, that there was an absolute beginning of the universe, i.e. a finite time in the past, when the universe was created, *ab initio*. Such proponents of an absolute beginning of time, influenced by the Biblical Scriptures, were the Christian theologians St. Augustine and Thomas Aquinas and the Hebrew theologian, Rambam (also known as Maimonides).

Maimonides and Aquinas

In the 13th century, Thomas Aquinas, a Christian theologian, agreed with Aristotle's idea of motion, as an absolute quality of matter.[2] He influenced the Roman Catholic Church to declare that the Earth does not move – that it is at rest at the center of the universe. This was the dispute between Galileo and the Pope several centuries later when the Holy Inquisition of the Roman Catholic Church put Galileo under house arrest for the remainder of his life for his heretical view that the Earth moves!

In the 12th century, the Hebrew theologian, Maimonides, asserted that motion *per se*, is not one of the primary attributes of matter.[3] As in Aristotle, he attributed motion to the accidents that matter undergoes in time, saying however, that it is the series of accidents that matter undergoes, as interactions, that are primary, rather than the effects of these accidents, that is its motion. He asserted that when God created the matter of the universe, *ab initio*, He created matter. Time, as a measure of the duration of matter, was one of its qualities.

Copernicus

In the 15th century, the astronomer Copernicus discovered that he could not explain the observations of the night sky, from one night to the next, by assuming a stationary Earth at rest in the center of the universe, with the Sun and the other 'planets' revolving about it. To explain his data, he postulated, rather, a heliocentric universe, with the Sun at the center, and the other planets, including the Earth, revolving about it. Thus, in disagreement with the Christian Church, Copernicus claimed that the empirical evidence indicated that the Earth moves! [4]

Galileo

A generation after Copernicus, Galileo's astronomical studies led him to agree with Copernicus that the Earth moves. But he enunciated his *principle of relativity*, asserting that the laws of nature are independent of the reference frame of the observer. That is, in contrast with Aristotle, Galileo said that 'motion' *per se* is a subjective ingredient in the laws of nature. Thus he went beyond Copernicus by saying that it is equally true to say that *from the Earth's perspective*, Sun revolves about Earth as it is to say that *from the Sun's* *perspective* Earth revolves about Sun! The perspective refers to the frame of reference from which observations are made. Thus Galileo superseded Copernicus in asserting that there is no absolute center of the universe.

Galileo's principle of relativity was an important precursor for *Einstein's principle of relativity*. An important difference is that in Einstein's theory the time measure is relative to the reference frame while in Galileo's theory the time measure is absolute – it is the same in all reference frames.

There are two types of variables in the laws of physics – the kinematic variables and the dynamic variables. The kinematic variables are the 'independent variables' while the dynamic variables are the 'dependent variables' – they are the solutions of the laws of nature. In classical dynamics of the particles of matter, the dependent variables are the locations of these particles, **r**(t), dependent on the time parameter t. Examples of kinematic relations in one dimensional space are:

$$x'(t) = x + Vt \qquad (1)$$

where V is the constant speed of one reference frame (unprimed), relative to the other (primed), x is the time- independent measure of a length x in the frame that moves relative to the observer and x'(t) is the measure of this length in the observer's frame.

If the object moves as well with a constant acceleration a in the x-direction, then Galileo deduced that the transformation of these lengths between the relatively moving reference frames is:

$$x'(t) = x + Vt + (1/2)at^2 \qquad (2)$$

It is important to emphasize that these kinematic relations between a spatial measure and the time measure are 'descriptive'; they are not 'explanatory'. Galileo's principle of relativity then affirms that the laws of nature are unchanged in form with respect to the transformations of space coordinates (1) (or (2)).

The explanation for the motion of matter comes from dynamical relations, for example Newton's second law of motion: if a body with inertial mass m is acted upon by an external force **F** (a cause) it will accelerate at the rate **a**, (an effect) relative to the observer, in a linear fashion. That is,

$$\mathbf{F} = m\mathbf{a}.$$

As discovered by Galileo, and Newton a generation later, this is a vector relation. That is, this equation represents three separate equations in 3-dimensional space. If $F_x = 0$ and $F_y \neq 0$, then $a_x = 0$ and $a_y \neq 0$. In the case of *Galileo's theory of gravity*, where $a_x = 0$ and $a_y = -g$, the kinematic rules for the composite motion in two dimensions is:

$$x'(t) = x + V_x t$$

$$y'(t) = y + V_y t - (1/2)gt^2$$

where -g is the constant acceleration due to gravity (in the downward direction, toward the center of the Earth) and (V_x, V_y) are the x- and y- time- independent components of the initial velocity **V**.

Galileo discovered that the combination of these two equations predicts projectile motion.

If there is no external force acting on a body, $\mathbf{F} = 0$, *then* $\mathbf{a} = 0$. This law asserts that if there is no external force acting on a body, then the motion of the body must be in a straight line at a constant velocity (or at rest). It is called '*Galileo's principle of inertia'*, or *Newton's first law of motion.*

Einstein's Theory of Special Relativity

The basis of Einstein's theory of special relativity entails a paradigm change from the classical roles of space, time and motion in the laws of nature.

The sequence of discoveries that led to the theory of relativity, as it was published in Einstein's 1905 paper, is as follows: In 1895, at the age of 16, after learning that Maxwell's equations entail the law of optics, Einstein asked this question: If light is supposed to propagate at the speed *c*, then what would it look like if it were at rest? That is to say, what is the solution of Maxwell's equations in a reference frame that would represent light to be at rest with respect to an observer? He then discovered the astounding result that there is no frame of reference that would describe the propagation of light at any speed other than the speed *c*. This seemed to defy common sense. For, according to common sense, if light were indeed a 'thing' that moves at the speed *c* relative to an observer, why would it not be possible for the observer to also move at the speed *c* next to the light beam, and then see it at rest?

Then Einstein realized that his non-common sense answer was based on a tacit assumption —that the law of nature (Maxwell's equations in this case) must keep its form in all possible reference frames relative to the observer. That is, one could come back to the common sense answer if it would be possible to change the form of the law of nature whenever one would change the frame of reference! But he saw that our common sense notions could very well be illusory, based on our sense reactions and our interpretations of them. Thus he concluded that the *objective* answer must be that the forms of the laws of nature must remain unchanged with any change of reference frame from that of an observer. This assertion is *the principle of covariance* (also called the *principle of relativity*). It underlies the theory of relativity (in its special or general forms).

Invariance of the Speed of Light

The principle of covariance is an assertion of the objectivity of the laws of nature – their independence of any reference frame in which they may be expressed, from the view of any particular observer. The role of the set of space and time measures is that of a language that is structured in order to express covariant laws of nature. In this regard it was found that a purely spatial measure in one reference frame must be expressed as a combination of spatial and temporal measures in other reference frames, $x' = \alpha x + \beta t$ where α and β are dimensionless coefficients that depend on the relative velocity of the reference frames.

Thus, the time measure must be expressed in the same units as the spatial measure. That is, instead of calling the temporal measure t sec, it is called ct cm, where c is a *conversion factor* with the dimension of cm/sec – a speed. In a different reference frame it would be called ct' cm, etc. This conversion factor must be universal – independent of reference frame, in order to preserve the physical role of the time parameter in the laws of nature. Thus we have predicted here that there must be an invariant speed that would appear in all covariant laws of nature. When examining the covariance of one particular law – Maxwell's equations for optics and electrodynamics - it turns out to be the speed of light in a vacuum. Since this speed is rooted in the spacetime language and is not particular to electromagnetism or optics, it is the same speed that must appear, as a conversion factor, in all other laws of nature.

We have seen, then, that the invariance of the speed of light c is a logical consequence of the principle of covariance. It is not a separate axiom of the theory of relativity. *The only axiom that underlies the theory of relativity is the principle of covariance.*

The Transformations of Space and Time Measures in Special Relativity

Because of the invariance of the speed of light to changes of reference frames, we have the following relation for a light beam that crosses Δx cm in Δt sec, in one reference frame, or $\Delta x'$ cm in $\Delta t'$ sec in another reference frame:

$$c = \Delta x/\Delta t = \Delta x'/\Delta t' \text{ cm/sec} \qquad (3)$$

This implies that the time measure as well as the space measure must change in the different reference frames, relative to an observer, to preserve the forms of the laws of nature. This was the first deviation from the Newtonian and Galilean law about absoluteness of the time measure.

In three-dimensional space, eq. (3) may be expressed in terms of the 4-dimensional Pythagorean rule:

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = \Delta s'^2 = c^2\Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2 = 0 \quad (4)$$

The theory of special relativity then says that not only for the laws of light, where the right side of (4) is zero, but also for all of the laws of matter, where the right hand-side of (4) is not zero, the invariance in the four-dimensional interval,

$$\Delta s = \Delta s' \qquad (4'),$$

must also hold. Then, what are the transformations that would mix the x and t coordinates (going back to one-dimensional space for convenience) to preserve the invariance in eq. (4')? To examine this problem, consider the Minkowski coordinate $x_4 = ict$ in the invariance

$$x^2 + x_4^2 = x'^2 + x_4'^2$$

The Lorentz Transformations

The general expression of the mixing of the space and time coordinates in a continuous fashion is a rotation

$$x' + ix_4' = (x + ix_4)\exp(i\varphi) = (x + ix_4)(\cos\varphi + i\sin\varphi)$$

where the primed coordinates are in the reference frame of the object that moves relative to the observer and the unprimed coordinates are those of the observer. Thus it follows that:

$$x' = x\cos\varphi - x_4\sin\varphi \qquad (5)$$

$$x_4' = x\sin\varphi + x_4\cos\varphi \qquad (6)$$

With constant relative speed between the reference frames at V cm/sec, $x'/x_4' = x'/ict' = -iV/c$. Assuming now that in a particular case, $x = 0$, we have, after dividing (5) by (6), $iV/c = \tan\varphi$. It then follows that

$$\cos\varphi = 1/[1 - (V/c)^2]^{1/2} \quad \text{and} \quad \sin\varphi = iV/c/[1 - (V/c)^2]^{1/2}$$

Thus, equations (5) and (6) become:

$$x' = (x + Vt)/[1 - (V/c)^2]^{1/2} \qquad (7)$$

$$t' = (t + (V/c^2)x)/[1 - (V/c)^2]^{1/2} \qquad (8)$$

These are the Lorentz transformations of the time measure and the spatial measure when one reference frame moves at the constant speed V relative to the other, in the x-direction.

The Fitzgerald-Lorentz Contraction

If the two ends of a spatial extension Δx are viewed at the same time $\Delta t = 0$, then eq. (7) becomes:

$$\Delta x = \Delta x' [1 - (V/c)^2]^{1/2} \qquad (9)$$

That is to say, the spatial measure in the observer's frame is less than the spatial measure in the moving frame of reference, $\Delta x < \Delta x'$. But this is nothing more than a *scale change* in the expression of the physical laws in the respective frames of reference. It does not at all refer to a physical change of a material body, such as the shortening of a meter stick that is in motion, by virtue of its motion relative to an observer. The latter 'physical change' would require dynamical laws of matter for their prediction. The *scale change*, on the other hand, is only a kinematic relation.

Simultaneity in Relativity Physics

Consider the Lorentz transformation (8) where t', t are intervals of time measure in the compared reference frames (Δt and $\Delta t'$), and x is a corresponding spatial interval Δx in the unprimed frame of reference.

We see here that a time difference in one frame my be zero ($\Delta t = 0$) while the corresponding time difference in the compared frame is not zero ($\Delta t' \neq 0$). That is to say, two events may be measured to be simultaneous in one reference frame but this measurement of events *is not seen to be* simultaneous in different reference frames that are in motion relative to the first one.

This signifies that the measures of the simultaneity of events, *in the language of a physical law*, are relative to the reference frame in which this law is expressed. It does not mean that a pair of events can *physically* happen simultaneously in one reference frame but not in another that is in motion relative to the first. For example, observing a pair of *physically* simultaneous events in a frame K (the proper frame) from a reference frame K' that is in motion relative to K, will make it *appear* to the observer in K' that the events are not simultaneous. However, in his or her attempt to correctly interpret the observation of the events, the observer transforms the language of space and time (using the Lorentz transformations) from his or her reference frame K', to the proper frame K of the events, and it is then discovered that indeed they are physically simultaneous events – *independent of any observer.*

Time Contraction

If the time interval between two events is measured at the same place, $\Delta x = 0$, eq. (8) becomes:

$$\Delta t = \Delta t'[1 - (V/c)^2]^{1/2} \qquad (10)$$

That is, the *measure* of a time change from the observer's frame in the frame of the moving object is less than the measure in the frame of the moving object itself, i.e. $\Delta t < \Delta t'$. As in the case of the Fitzgerald-Lorentz contraction of the measures of length, this does not signify a physical change of duration in one frame of reference compared with the other. That is, it does not signify the change of the cell decay rate in the biological aging of a human being in a frame that is in motion compared with the aging of an observer of this human being in motion. Nor does it signify a slowing down of the hands of a clock in the moving frame compared with the clock in the observer's frame. This is not more than a *scale change* of the measure of duration in the moving frame compared with the frame of the observer's clock. For example, to describe the laws of nature correctly in the moving frame the observer may have to put eight digits on the face of the moving clock, rather than the twelve digits on his own clock. But this does not mean that anything physical has happened to the spring behind the face of the moving clock that does not happen to the spring of the observer's clock! Physical laws of nature of material systems must predict physical changes, not kinematic relations, as are the Lorentz transformations. The paradoxes in relativity theory have arisen (e.g. the twin paradox) when this false interpretation is used in calling the kinematic relations physical relations.

The Twin Paradox

Consider the twin paradox. If we should (mistakenly) interpret the transformation of the time measure (10) as a physical change, then if a twin brother Peter moves away from his brother Paul on a round trip, he would be younger than Paul when he returns. But from the view of Peter, it is Paul who is moving away from Peter and would be younger when he returns. This is a logical paradox since it implies that after the round trip of one of the brothers relative to the other, he would be both older and younger than his brother. But when we recognize the subjectivity of motion (as Galileo discovered) and that the transformation of time measures is only a scale change, used by one or the other brother in viewing his moving brother, there is no paradox because there is no prediction of a change in the aging of a body by virtue of its motion relative to an observer.[7]

Transformations of Velocities in Special Relativity

If we consider the measures of space and time in terms of differential changes, then dividing eq. (7) by eq. (8) gives:

$$v_x' = dx'/dt' = ([dx + Vdt]/[1 - (V/c)^2]^{1/2})([1 - (V/c)^2]^{1/2}/[dt + (V/c^2)dx])$$
$$= [dx/dt + V]/[1 + v_xV/c^2] = [v_x + V]/[1 + v_xV/c^2] \qquad (11)$$

where V is the constant relative speed between the reference frames in the x-direction.

We see, then, in (11), that if the object moves at the speed $v_x = c$ in the moving frame of reference, then in the observer's frame,

$$v_x' = [c + V]/[1 + V/c] = c[1 + V/c]/[1 + V/c] = c$$

This was Einstein's non-common sense finding: that the observer's measure of the speed of an object c, in a reference frame that moves relative to the observer at the speed c, is also equal to c (it is not equal to 2c!) For example, in classical physics, if an object is in motion at c cm/sec, relative to the floor of a train that moves relative to an observer at v cm/sec, then the stationary observer would expect to see the object move at (c + v) cm/sec. But relativity theory predicts that this is not true, that he would see it move at c cm/sec! Yet, this non-common sense prediction of relativity theory is a consequence of *the principle of covariance* – the assertion that the laws of nature are objective, that they must have the same form in all possible frames of reference.

The Meaning of the Invariance of c and Causality in Relativity Physics

The question continually arises: What is the physical reason in relativity theory that one cannot cause an object to move faster than the speed of light? The answer is that the actual role of the speed *c* in the theory of relativity is that it is the maximum speed of propagation of a force (of any type) between an emitter (the cause of motion of another body) and an absorber (the effect that is its motion). Thus if an object could move faster than the speed of light, the force that causes its motion could not catch up with it in order to cause its continued motion. Thus, according to the theory of relativity, no object, matter or radiation, can propagate faster than the speed of light in a vacuum, *c,* relative to any observer.

It is interesting to note that the time measure is used here to parameterize the causal connection between interacting bodies. A signal propagates at the speed *c*, from one end of an interaction – the emitter E, which is the cause of an effect later on – to the other end of the interaction – the effected absorber A – *in a finite time*. (This is in contrast with classical physics, wherein there is *action at a distance*, where the interaction between distant bodies is spontaneous). If R', the spatial separation between A' and E, is greater than the distance traveled by the interacting signal, R = ct, at c cm/sec, in t seconds, then there can be no interaction between A' and E, i.e. *they cannot be causally connected*. In this sense, then, the time measure t may be defined in terms of a causal connection between interacting bodies that are $R \leq ct$ cm apart.

In classical physics, the time measure *t* is absolute (independent of any reference frame in which the interaction is expressed). Thus, the concept of causality in classical physics is similarly absolute. In Einstein's relativity theory, on the other hand, causality is a relative concept since the time measure, as a parameterization of the cause-effect relation, is a function of the reference frame in which it is expressed, in the language of the laws of nature.

Concluding Remarks

We have seen that a paradigm change in the theory of relativity occurs in the expression of the 'motion' of objects. In classical physics the velocity of motion of an object is denoted by the vector, $\mathbf{v} = d\mathbf{r}/dt$. In special relativity, in a different reference frame the numerator and the denominator of this form become a mixture of differential amounts of space and time changes. Thus, to preserve covariance, $d\mathbf{r}/dt \rightarrow d\mathbf{r}'/dt'$, - both the numerator and the denominator become mixtures of differential amounts of spatial and temporal change in the other (unprimed) reference frames. Thus the parameters of 'motion' are not absolute in time or space. The only invariants are the forms of the laws of nature and the differential in 4-dimensional space, ds, eq. (4'). It is the set of transformations that leave ds invariant that underlie the form invariance of the laws, (their covariance) in special relativity theory.

In the field theory implied by Einstein's theory of relativity, the space and time parameters are not absolute features of matter. They are instead 'independent variables'. The 'fields' that represent nature (as are, for example, the solutions of Maxwell's equations) are the 'dependent variables', mapped into the space of the independent variables. Thus, 'motion' in relativity theory is *a subjective feature of objective laws of nature*. What it is that is moving in this view is not a singular particle of matter, but rather a 'mode' of a matter continuum, analogous to the ripple of a pond.

General Relativity

To this point, we have been discussing aspects of the theory of relativity that refers to 'special relativity'. This is based on the covariance of the field laws with respect to a very special type of relative motion - reference frames that are in motion at a constant speed in a straight line – inertial frames of reference. The generalization of this theory recognizes that *a force that acts on a body originating in another body causes non-inertial 'motion'*. The relative motion of any body is then not generally inertial. In this case, the generally variable distribution of matter in the universe causes a body to move at speeds that are not constant in time.

In this case of non-inertial motion of matter – the natural motion of any matter - the invariance of the (squared) differential metric in special relativity, $(ds^2)_{sr} = c^2dt^2 - dx^2 - dy^2 - dz^2$, characterized by the constant 'signature (1, -1, -1, -1), must be replaced by a continuously variable signature, dependent on the metric tensor $g_{\mu\nu}(x)$ in

$$ds^2 = g_{00}(x)dx^{0\,2} + g_{11}(x)dx^{1\,2} + \ldots + g_{01}(x)dx^0\,dx^1 + .. = \Sigma_{\mu\nu}g_{\mu\nu}(x)dx^\mu dx^\nu$$

where '0' denotes the time coordinate $x^0 = ct$, and $\mu, \nu = 0, 1, 2, 3$ refer to the time and space coordinates.

$g_{\mu\nu}(x) = g_{\nu\mu}(x)$ is the ten-component symmetric metric tensor. It is a continuous function of the spacetime coordinates x. Such a spacetime is called "Riemannian". In the limit of a matterless universe, it approaches $(ds^2)_{sr}$ In this limit, the components of the metric tensor approach the signature of the special relativity metric, $g_{00} \rightarrow 1$, $g_{kk} \rightarrow -1$ (k = 1, 2, 3) and $g_{\mu\neq\nu} \rightarrow 0$. The latter ideal situation is the (in principle) unreachable *flat* spacetime limit, though it is asymptotically approachable, as an approximation. The limit corresponds to the single bit of matter in a perfect vacuum.

The dynamics in general relativity is based on the *Riemannian geometry* wherein spacetime is 'curved', *everywhere*. This curvature, in turn, leads to the phenomenon of gravity. This is analogous to the curvature of the Earth, along its entire surface. The ideal limit of a matterless universe, described by the special relativity metric $(ds^2)_{sr}$, is the *flatspace* characterized by *Euclidean geometry*.

It can be shown that indeed the dynamics of matter in the field theory of general relativity must unify all of the forces in nature, including gravity, in terms of a generalized curved spacetime that, in turn, represents the presence of matter *anywhere* in the universe, from the cosmological domain to that of elementary particles.[8] It is this approach that will likely explain the physics of the future, as we proceed into the 21st century, in terms of laws of nature (including the formal expression of Quantum Mechanics) that are rooted in Einstein's theory of general relativity.

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